

April 2, 2008 added Appendix B: SU(2)

Analytic Functions of a Quaternionic Variable

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Abstract

Here we follow the basic analysis that is common for real and complex variables and find how it can be applied to a quaternionic variable. Non-commutativity of the quaternion algebra poses obstacles for the usual manipulations; but we show how many of those obstacles can be overcome. After a tiny bit of linear algebra we look at the beginnings of differential calculus. The surprising result is that the first order term in the expansion of $F(x + \delta)$ is a compact formula involving both $F'(x)$ and $[F(x) - F(x^*)]/(x - x^*)$.

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1 Introduction

We are very familiar with functions of a real or complex variable x which we can expand, in the mode of differential calculus, as

$$F(x + \delta) = F(x) + F'(x)\delta + \frac{1}{2}F''(x)\delta^2 + \dots \quad (1.1)$$

But what if we consider a quaternionic variable

$$x = x_0 + ix_1 + jx_2 + kx_3, \quad (1.2)$$

where those quaternions i, j, k do not commute with one another. The small quantity δ will also involve all those quaternions. How then can we expect anything as neatly packaged as Equation (1.1)?

This is a long-standing challenge to mathematicians; and here we believe we have something new to offer. Previous work has attempted to extend the usual concepts of the derivative, $\frac{dF}{dx}$: see, for example, references [1], [2] and [3]. We shall, instead, focus only on the differential dF , as suggested by Eq. (1.1). The method will be to start by examining the exponential function e^x , in Section 3, and then use this result to build more general functions $F(x)$, in Section 4. We succeed in finding a general formula for the first order term that is surprisingly compact, as shown in Eq. (4.3); but the price we pay is that it is no longer local, involving $[F(x) - F(x^*)]/(x - x^*)$ along with $F'(x)$.

In Section 5 we describe a geometric view of this situation; in Section 6 we find the Leibnitz rule to be valid. Section 7 presents the second order term in the generalized expansion (1.1) and Section 8 contains a general discussion of these results. In Appendix B we show how this line of analysis can be extended to the non-commutative variables based on the Lie algebra $SU(2, C)$.

First, however, we make a brief visit to quaternionic linear algebra.

2 Linear Algebra

Suppose we have the linear equation

$$ax + xb = c \quad (2.1)$$

where a, b, c are given quaternions and we want to solve for the unknown quaternion x . If we were dealing with real or complex quantities, then all the

factors in Equation (2.1) would commute and we would write $x = (a+b)^{-1} c$. But that does not work for quaternionic numbers and variables.

In fact, the solution to Equation(2.1) may be written as

$$x = d^{-1}(ac + cb^*), \quad d = a^2 + a(b + b^*) + b^*b \quad (2.2)$$

where one recognizes that $[a, d] = ad - da = 0$. The symbol $*$ means the usual complex conjugation, which changes the signs of all the imaginaries. I would imagine that this solution is not new but I do not know where to find that out.

There is an alternative way of writing the solution:

$$x = (a^*c + cb)h^{-1}, \quad (2.3)$$

which I will leave for the reader to explore.

3 Exponential Function

For a general quaternionic variable x , we can define the exponential function in the usual way

$$e^x = \lim_{N \rightarrow \infty} \left(1 + \frac{x}{N}\right)^N \quad (3.1)$$

and this leads us to the expansion

$$e^{(x+\delta)} = e^x \left[1 + \int_0^1 ds e^{-sx} \delta e^{sx} + O(\delta^2)\right]. \quad (3.2)$$

This formula is well known to some people (I believe it is often credited to Richard Feynman but I do not know a reference); a derivation of it is given in Appendix A.

If we now separate x into real and imaginary parts as $x = x_0 + ru_x$, where $(u_x)^2 = -1$, and, furthermore, use the expansion

$$e^x = e^{x_0} (\cos r + u_x \sin r), \quad (3.3)$$

then we can write the expansion as

$$e^{(x+\delta)} = e^x + e^x [a \delta + b [u_x, \delta] + c u_x \delta u_x + O(\delta^2)] \quad (3.4)$$

where the real quantities a, b, c are given by

$$a = \frac{1}{2} \left(1 + \frac{\sin 2r}{2r}\right), \quad b = \frac{1}{2} \frac{\cos 2r - 1}{2r}, \quad c = \frac{1}{2} \left(-1 + \frac{\sin 2r}{2r}\right). \quad (3.5)$$

In the special case where u_x commutes with δ , then this reduces to the familiar formula for the exponential of a complex variable.

We can now go on to the logarithm function, defined as the inverse of the exponential:

$$e^x = y, \quad x = \ln y, \quad e^{(x+\delta)} = y + \Delta, \quad x + \delta = \ln(y + \Delta). \quad (3.6)$$

Using the previous results, we eventually arrive at the expansion

$$\begin{aligned} \ln(y + \Delta) &= \ln y + Ay^{-1}\Delta + B[u_x, y^{-1}\Delta] + Cu_x y^{-1}\Delta u_x + O(\Delta^2), \\ A &= \frac{1}{2}(r \cot r + 1), \quad B = \frac{r}{2}, \quad C = \frac{1}{2}(r \cot r - 1). \end{aligned} \quad (3.7)$$

4 General Function $F(x)$

For a general function $F(x)$ of a quaternionic variable $x = x_0 + ru_x$, we start by assuming a representation as a Laplace transform:

$$F(x) = \int dp f(p) e^{px} \quad (4.1)$$

where p is a real variable. We then calculate an expansion for $F(x + \delta)$ using the formula derived above for the exponential function; and this leads to the following general formula

$$\begin{aligned} F(x + \delta) &= F(x) + F'(x) \delta + \frac{1}{4r} (-F(x) + F(x^*)) [u_x, \delta] + \\ &\quad \frac{1}{4} F'(x) [u_x, [u_x, \delta]] + O(\delta^2), \end{aligned} \quad (4.2)$$

where $F'(x)$ is the usual derivative of the function $F(x)$ calculated as if x were a real variable. It is surprising how simple this formula appears.

Another form of this formula (4.2) is

$$F(x + \delta) - F(x) = F'(x) \delta_1 + (F(x) - F(x^*)) (x - x^*)^{-1} \delta_2 + O(\delta^2), \quad (4.3)$$

$$\delta_1 = \frac{1}{2} (\delta - u_x \delta u_x), \quad \delta_2 = \frac{1}{2} (\delta + u_x \delta u_x), \quad (4.4)$$

which will be discussed further in the next Section.

If one considers the function $F(x) = x^n$, then the above formula, (4.2), is correct, as may be shown by induction. Thus it is also true for any power series, $F(x) = \sum_n c_n x^n$.

It is an interesting exercise to show that the special formula (3.7) for the logarithm function is in agreement with the general formula (4.2).

5 Geometric View

The variable x , as written in Eq. (1.2), may be viewed as a point in a four-dimensional space composed of the real line, for x_0 , and a three-dimensional Euclidean space, for the vector $\mathbf{x} = (x_1, x_2, x_3)$. In the decomposition $x = x_0 + ru_x$, we see r as the length of the vector $\mathbf{x} = r\hat{x}$; and the unit imaginary quaternion u_x is the dot product of the unit vector \hat{x} with the vector (i, j, k) .

Now we want to recognize that the quaternion δ , which is used to displace the variable x when we write $F(x + \delta)$, can be given a different decomposition in that four-dimensional space: as defined in the previous section, Eq. (4.4), $\delta = \delta_1 + \delta_2$. We can recognize that δ_1 is in the two-dimensional space composed of the real line and the radial direction along the vector \mathbf{x} ; and δ_2 is in the two-dimensional space that is orthogonal to the direction of \mathbf{x} . One can say, geometrically, that δ_1 is parallel to x (perhaps writing it as δ_{\parallel}) and δ_2 is perpendicular (δ_{\perp}). Algebraically, this is represented by the relations,

$$\delta_1 x = x \delta_1, \quad \delta_2 x = x^* \delta_2. \quad (5.1)$$

This decomposition of δ is a local process. It varies from one place x to another, rather like the familiar unit vectors in polar coordinates of two-dimensional Euclidean space.

The authors of reference [1] have taken a similar approach, introducing a local unit imaginary (which they call *iota*) that is the same as what we have defined as u_x . However, they limit their differentiations to displacements that are restricted to this two-dimensional space: they allow only what we call δ_1 without any of δ_2 . In that way they merely reproduce what is known about ordinary complex variables.

6 Leibnitz' Rule

Let us now define the first-order differential operator \mathcal{D} , from Eq.(4.3), as

$$F(x + \delta) = F(x) + \mathcal{D} F(x) + O(\delta^2) \quad (6.1)$$

with

$$\mathcal{D} F(x) = F'(x) \delta_1 + (F(x) - F(x^*)) (x - x^*)^{-1} \delta_2. \quad (6.2)$$

Now we consider $\mathcal{D}(F(x)G(x))$. It starts off easy:

$$\mathcal{D}(FG) = (F(x) + \mathcal{D}F)(G(x) + \mathcal{D}G) - FG = F\mathcal{D}G + (\mathcal{D}F)G; \quad (6.3)$$

but the next step is more interesting, substitution (6.2) into (6.3):

$$\begin{aligned}\mathcal{D}(FG) = & F[G'(x) \delta_1 + (G(x) - G(x^*)) (x - x^*)^{-1} \delta_2] + \\ & [F'(x) \delta_1 + (F(x) - F(x^*)) (x - x^*)^{-1} \delta_2] G. \quad (6.4)\end{aligned}$$

The question is, How do we move the quaternions δ to the right, past the function G in the second term of Eq. (6.4)? The answer is given by (5.1); and we finally get,

$$\mathcal{D}(FG) = (F G' + F' G) \delta_1 + [F(x)G(x) - F(x^*)G(x^*)](x - x^*)^{-1} \delta_2, \quad (6.5)$$

which corroborates Leibnitz' rule for this differential operator.

7 Second Order Terms

Let's return to the exponential function (3.1) and proceed with the expansion,

$$e^{(x+\delta)} = e^x [1 + \int_0^1 ds e^{-sx} \delta e^{sx} + \int_0^1 dt \int_0^{1-t} ds e^{-(s+t)x} \delta e^{tx} \delta e^{sx} + O(\delta^3)]. \quad (7.1)$$

The better approach is to combine the exponential function and the Laplace transform from the beginning. Writing $F(x + \delta) = F(x) + F^{(1)} + F^{(2)} + \dots$, we will first re-do the calculation of $F^{(1)}$ to see how easily it goes with the decomposition $\delta = \delta_1 + \delta_2$ and the algebra (5.1).

$$F^{(1)} = \int dp f(p) p \int_0^1 ds e^{(1-s)px} (\delta_1 + \delta_2) e^{spx} = \quad (7.2)$$

$$\int dp f(p) p e^{px} \int_0^1 ds [\delta_1 + e^{sp(x^*-x)} \delta_2] \quad (7.3)$$

and this leads immediately to the result Eq. (4.3).

Now we look at

$$F^{(2)} = \int dp f(p) e^{px} p^2 \int_0^1 dt \int_0^{1-t} ds e^{-(s+t)px} \delta e^{tpx} \delta e^{spx}. \quad (7.4)$$

Again, we decompose δ and after a bit more work arrive at the result for the second order term,

$$\begin{aligned}F^{(2)} = & \frac{1}{2} F''(x) \delta_1^2 + (F(x) - F(x^*)) (x - x^*)^{-2} (\delta_2 \delta_1 - \delta \delta_2) + \\ & F'(x) (x - x^*)^{-1} \delta \delta_2 + F'(x^*) (x^* - x)^{-1} \delta_2 \delta_1. \quad (7.5)\end{aligned}$$

8 Discussion

One may ask what restrictions there are on the functions $F(x)$ considered above. At first, one would say that they should be real analytic functions; having terms like axa where a is a general quaternion would certainly cause trouble. One can extend this condition slightly by allowing $F(x)$ (but not the function $G(x)$ in Section 6) to be a real function with arbitrary quaternions multiplying from the left. That is, the power series form $F = \sum_n c_n x^n$ could have arbitrary numbers c_n .

This bias to the left-hand side can be reversed if we change the original steps (3.2), setting $s \rightarrow 1 - s$, and (4.1), putting $f(p)$ on the right-hand side.

It is noteworthy that our differential operators are no longer local: they involve $F(x^*)$ along with $F(x)$. Nevertheless, it is surprising how simple and how general the results obtained here are.

The Taylor series we have discussed above are expansions about the origin $x = 0$. In the usual complex analysis such power series may be about any fixed point $x = x_f$; but such a quaternion constant put in the middle of our expressions would appear to cause trouble. That can be rectified by defining the new variable $y = x - x_f = y_0 + r_y u_y$ and then using this new imaginary u_y to separate the displacement $\delta = \delta_1 + \delta_2$.

One may also ask if this general method may be applied to some other kind of non-commuting algebra beyond the quaternions. I believe that something very similar can be done starting with a Clifford algebra. Another example is given in Appendix B.

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Appendix A

Here we give a derivation of the formula (3.2) for any non-commuting quantities x and δ .

$$e^{(x+\delta)} = \lim_{N \rightarrow \infty} [1 + \frac{x}{N} + \frac{\delta}{N}]^N = \quad (\text{A.1})$$

$$\lim_{N \rightarrow \infty} \{ [1 + \frac{x}{N}]^N + \sum_{m=0}^{N-m-1} [1 + \frac{x}{N}]^{N-m-1} \frac{\delta}{N} [1 + \frac{x}{N}]^m + O(\delta^2) \}. \quad (\text{A.2})$$

In taking the limit $N \rightarrow \infty$, we convert the sum over m to an integral over $s = \frac{m}{N}$ and this yields

$$e^{(x+\delta)} = e^x + \int_0^1 ds e^{(1-s)x} \delta e^{sx} + O(\delta^2). \quad (\text{A.3})$$

Appendix B

Here we shall extend the general method used above for a quaternionic variable to something built on a more general Lie Algebra - specifically $SU(2)$.

Here is the Lie algebra:

$$[J_1, J_2] = J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2, \quad (\text{B.1})$$

where the three J 's are understood to be matrices over the complex numbers. In particular we shall use the relations

$$e^{\theta J_3} J_1 e^{-\theta J_3} = J_1 \cos\theta + J_2 \sin\theta, \quad e^{\theta J_3} J_2 e^{-\theta J_3} = J_2 \cos\theta - J_1 \sin\theta, \quad (\text{B.2})$$

which follow from (B.1).

The new variable x is to be constructed with four real parameters as

$$x = x_0 I + x_1 J_1 + x_2 J_2 + x_3 J_3 \quad (\text{B.3})$$

and we want to expand $F(x + \delta) = F(x) + F^{(1)} + O(\delta^2)$, where δ is a small quantity in that same space of matrices as x . Our first step is to define a local coordinate system at the given point x . By a suitable linear transformation (rotation) of the Lie algebra we make the coordinate x appear as

$$x = x_0 I + r J_3 \quad (\text{B.4})$$

where we recognize that $r^2 = x_1^2 + x_2^2 + x_3^2$.

We can now separate the displacement $\delta = \delta_{\parallel} + \delta_{\perp}$ as follows.

$$\delta_{\parallel} = \delta_0 I + \delta_3 J_3, \quad \delta_{\perp} = \delta_1 J_1 + \delta_2 J_2. \quad (\text{B.5})$$

Now we are ready to study the first order term in the expansion, again using the representation of $F(x)$ in terms of the exponential function.

$$F^{(1)} = \int dp f(p) p e^{px} \int_0^1 ds e^{-spx} \delta e^{spx}. \quad (\text{B.6})$$

Since δ_{\parallel} commutes with x , the first part of this is simply $F'(x) \delta_{\parallel}$. For the part with δ_{\perp} we use the formulas (B.2), where θ is replaced by $-spr$. The integrals over s are trivial and we merely write $\sin(pr)$ and $\cos(pr)$ in terms of $e^{\pm i pr}$ to get our final result.

$$F(x + \delta) = F(x) + F'(x)\delta_{\parallel} + \{F(x + ir) - F(x - ir)\} \frac{1}{2ir} \delta_{\perp} + \{F(x + ir) + F(x - ir) - 2F(x)\} \frac{1}{2r} [J_3, \delta_{\perp}] + O(\delta^2). \quad (\text{B.7})$$

It should be noted that the δ -related factors in Eq. (B.7) can be written in the following way:

$$[J_3, \delta_{\perp}] = \frac{1}{r} [x, \delta], \quad (\text{B.8})$$

$$\delta_{\perp} = -\frac{1}{r^2} [x, [x, \delta]], \quad (\text{B.9})$$

$$\delta_{\parallel} = \delta - \delta_{\perp}. \quad (\text{B.10})$$

This means that we do not have to carry out the "rotation" that gave us Eq. (B.4) explicitly; the talk about choosing a local coordinate system is merely rhetorical.

I expect that this method can be extended to other Lie algebras, with the quantity δ_{\perp} subdivided into distinct portions according to the roots of the particular algebra. The system of Eqs. (B.9), (B.10) would be adapted to make those separations, using the known values of the roots; and those root values would also appear in the final generalization of Eq. (B.7).

References

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